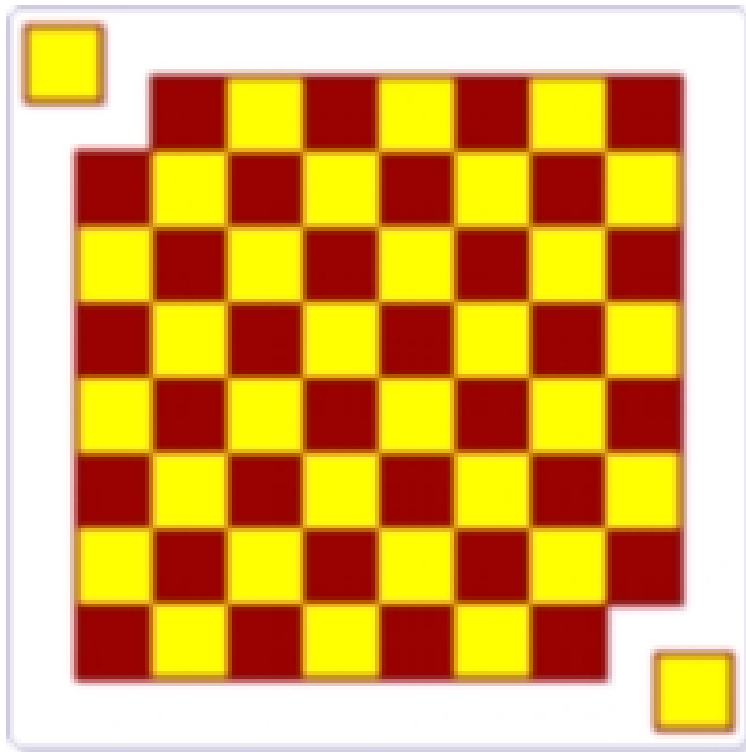# MATHEMATICS ON THE CHESSBOARD

Problem 1. Consider a 8×8 chessboard and remove two diametrically opposite corner unit squares. Is it possible to cover (without overlapping) the remaining 62 unit squares with dominoes?
(A domino is a 1×2 rectangle).

Solution. Two diametrically opposite corner squares that have been removed from the original chessboard have the same color. Clearly, then, since a domino covers one white unit square and one black square, it is impossible to cover the remaining 62 unit squares with dominoes.

Problem 2. In each unit square of a 8×8 array we write one of the numbers -1, 0 or 1. Is it possible that all sums on rows, columns and the two diagonals are distinct?

Solution. No! We have $2 \cdot 8 + 2 = 18$ sums. The maximum value of such a sum is 8 and its minimum value is -8. Therefore the 18 numbers lie in the set

$$\{-8, -7, \ldots, 0, 1, \ldots 8\}.$$

Since the above set contains exactly 17 numbers, at least two of the above sums must be equal.

Problem 3. (a) Is it possible to fill the unit squares of a 7×7 array with 1 or -1 such that the product of the elements in each row is 1 and the product in each column is -1?

(b) What is we consider a 8×8 board? In how many ways ?

Solution. Denote by $a_1, a_2, \ldots a_7$ the product of the elements in each row and by $b_1, b_2, \ldots b_7$ product of the elements in each column. Then $a_1 a_2 \cdots a_7$ and $b_1 b_2 \cdots b_7$ represent the product of all the elements in the array, so the must be equal. According to our condition we have

$$a_1 a_2 \cdots a_7 = 1$$

while

$$b_1 b_2 \cdots b_7 = (-1)^7 = -1$$

which is a contradiction.

(b) For a 8×8 array, the above argument does not lead to any contradiction. Remark that the on column 1, the first 7 unit squares can be filled in 2^7 different ways (as each entry must be either 1 or -1) and the last unit square can be filled in only one way. Similarly, columns 2,3,4,5,6 and 7 can be filled in 2^7 ways each. For the last column, the number of 1 or -1 written in each of its unit squares are uniquely determined by the product on rows which must be 1. Therefore the final answer is $(2^7)^7 \cdot 1 = 2^{14}$.

Problem 4. Each unit square of a 25×25 board is filled with 1 or -1. Denote by $a_1, a_2, \ldots, a_{25}$ the products of the elements by rows and by $b_1, b_2, \ldots, b_{25}$ the product of the elements by columns. Prove that

$$a_1 + a_2 + \cdots + a_{25} + b_1 + b_2 + \cdots + b_{25} \neq 0.$$

Solution.

Remark first that $a_1 a_2 \cdots a_{25}$ and also $b_1 b_2 \cdots b_{25}$ represents the product of all the numbers on the chessboard. Therefore,

(1) $$a_1 a_2 \cdots a_{25} = b_1 b_2 \cdots b_{25}.$$

Denote by k (resp. p) the number of -1 between a_1, a_2, $\ldots$ a_{25} (resp. b_1, $b_2, \ldots b_{25}$). Then (1) reads

$$(-1)^k = (-1)^p,$$

that is, k and p have the same parity. Now

$$
\begin{aligned}
a_1 + a_2 + \cdots + a_{25} + b_1 + \cdots + b_{25} &= (25 - k) - k + (25 - p) - p \\
&= 2(25 - k - p) \\
&= 2[25 - (k + p)]
\end{aligned}
$$

which is never zero since $k + p$ is even (why?)

Problem 5. Seven unit cells of a 8×8 chessboard are infected. In one time unit, the cells with at least two infected neighbours (having a common side) become infected. Can the infection spread to the whole square?

Solution. By looking at a healthy cell with 2,3 or 4 infected neighbors, we observe that the perimeter of the infected area does not increase. Initially the perimeter of the contaminated area is at most $4 \times 7 = 28$ so it never reaches $4 \times 8 = 32$. Therefore, the infection cannot spread to the whole chessboard.
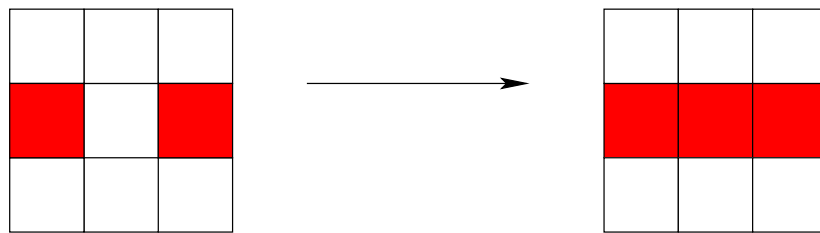
Figure 1. A normal cell (white) having two infected neighbours
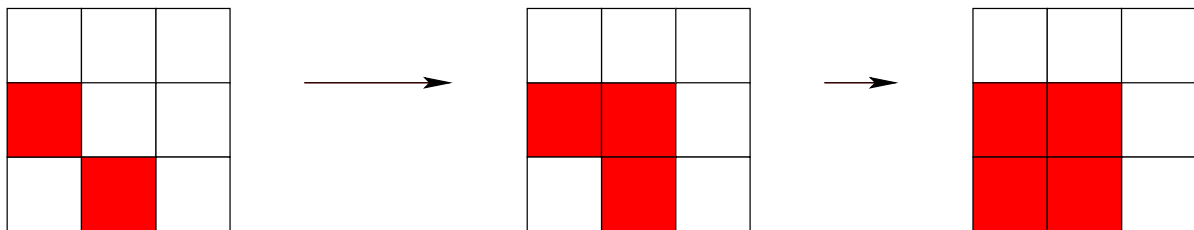
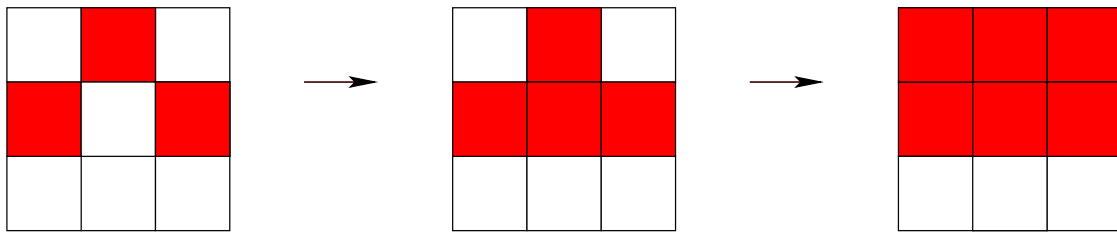Figure 2. A normal cell (white) having two infected neighbours

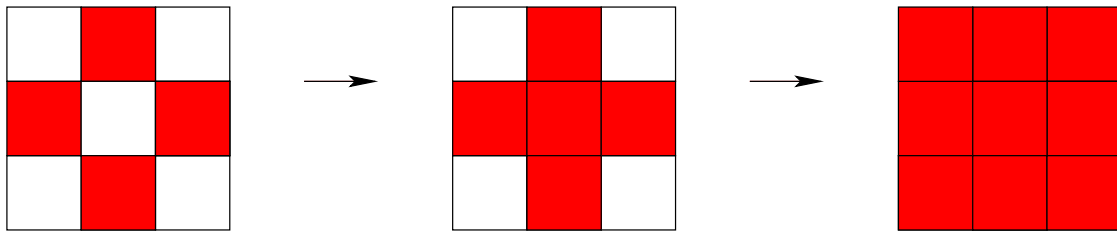Figure 3. A normal cell (white) having three infected neighbours

Figure 4. A normal cell (white) having four infected neighbours

Similar variant. Initially, some configuration of cells of a given $n \times n$ chessboard are infected. Then, the infection spreads as follows: a cell becomes infected if at least two of its neighbors are infected. If the entire board eventually becomes infected, prove that at least n of the cells were infected initially.

Problem 6. The numbers $1, 2, \ldots, 81$ are randomly written in a 9×9 array. Prove that there exists a 2×2 subarray whose numbers have the sum greater than 137.

Solution. There are exactly $8 \cdot 8 = 64$ subarrays of type 2×2.

Problem 6. The numbers $1, 2, \ldots, 81$ are randomly written in a 9×9 array. Prove that there exists a 2×2 subarray whose numbers have the sum greater than 137.

Solution. There are exactly $8 \cdot 8 = 64$ subarrays of type 2×2.
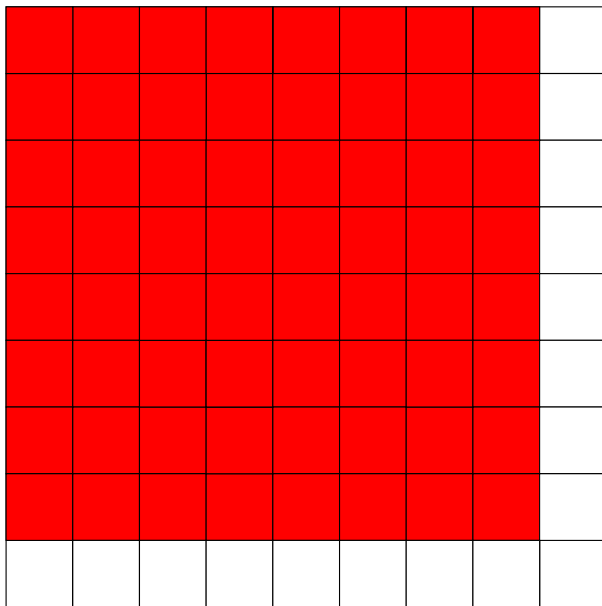

Figure 5. The top left unit square of any 2×2 must be one of the red squares

Let

$$S_1 \leq S_2 \leq \cdots \leq S_{64}$$

be the sums of numbers written in these subarrays. Suppose that the assertion of the problem does not hold, that is, the largest of the sums in question satisfies the inequality $S_{64} \leq 137$. This also implies

$$S_1 + S_2 + \cdots + S_{64} \leq 64 \cdot 137 = 8768.$$

On the other hand, in the above sum some of the numbers in the array are counted exactly once, some others are counted twice and some of them are counted four times.
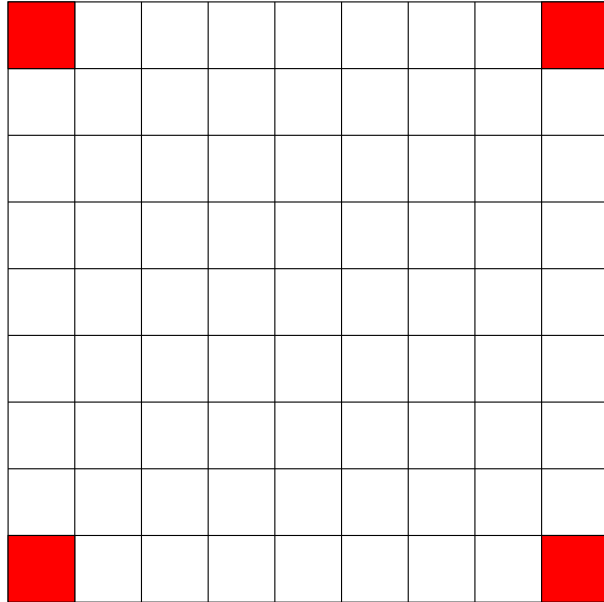

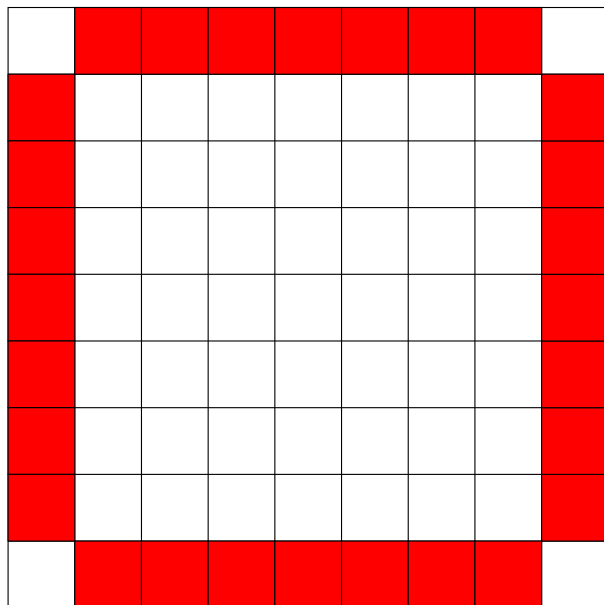Figure 6. The numbers written in the red unit squares are counted only once

Figure 7. The numbers written in the red unit squares are counted exactly twice
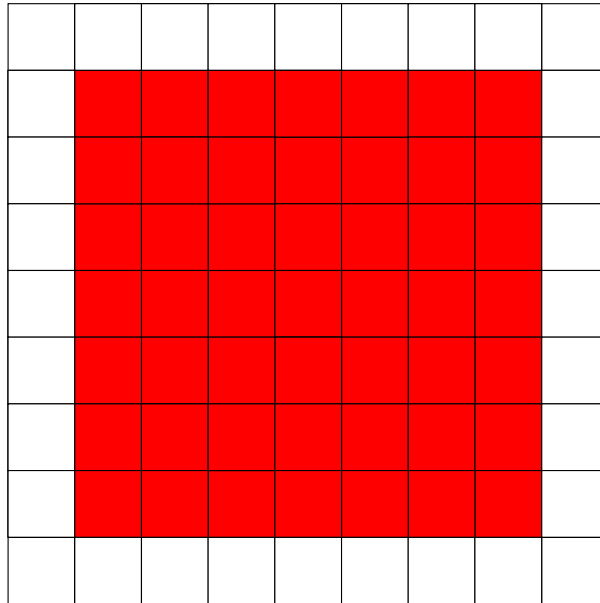
Figure 8. The numbers written in the red unit squares are counted exactly four times

We have therefore the lower bound

$$
\begin{aligned}
S_1 + S_2 + \cdots + S_{64} \ \geq \ & 1(81 + 80 + 79 + 78) + 2(77 + 76 + \cdots + 50 \\
& + 4(49 + 48 + \cdots + 1) \\
= \ & 8774,
\end{aligned}
$$

contradiction. Therefore, at least one of the sums in the 2×2 subarray is greater than 137.

Problem 7. In how many ways is it possible to fill the unit squares of a chessboard with -1 and 1 such that the sum of elements in each 2×2 subarray is 0? (Columbia Math Olympiad)

Solution. The first column can be filled in exactly 2^8 ways. If the numbers 1 and -1 in the first column alternate (we have two ways in this case) then the second column is either equal to the first one or exactly opposite to it. Hence we have two ways to fill each of the columns 2,3,...,8.

If the numbers 1 and -1 in the first column do not alternate (we have $2^8 - 2$ possibilities for the first column in this case) then there exist two adjacent unit squares in which it is written the same number, say 1. Then, in the next two squares on the second column we must have -1. Therefore the second column is completely determined, so are the next columns.
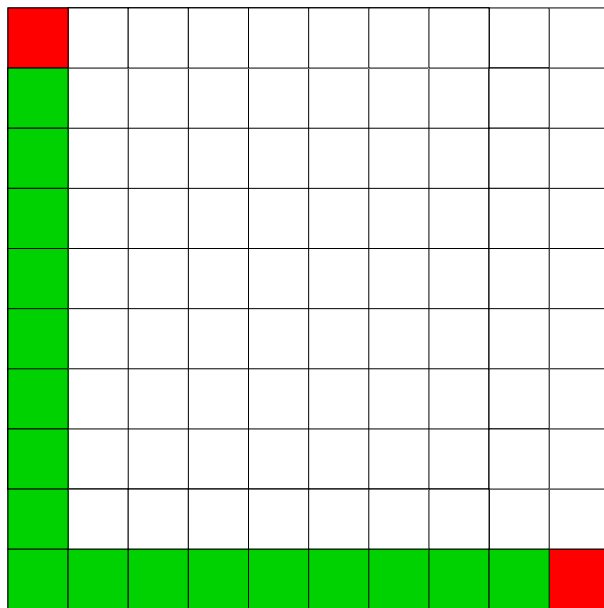The total number is

$$2^8 + (2^8 - 2) = 2^9 - 2 = 2046.$$

Problem 8. The numbers $1, 2, \ldots, 100$ are randomly written in a 10×10 array. Prove that there exists two neighbouring unit squares (sharing a side in common) such that the numbers x, y written in them satisfy $|x - y| \geq 6$.

Solution. Assume that the conclusion in the above statement does not hold. Then, the absolute value of the difference of the numbers written in any two neighbouring unit squares is at most 5. The largest and the smallest numbers on the board are 1 and 100. They can be joined by a chain of at most 19 neighbours unit squares as depicted in the figure below.

Problem 8. The numbers $1, 2, \ldots, 100$ are randomly written in a 10×10 array. Prove that there exists two neighbouring unit squares (sharing a side in common) such that the numbers x, y written in them satisfy $|x - y| \geq 6$.

Solution. Assume that the conclusion in the above statement does not hold. Then, the absolute value of the difference of the numbers written in any two neighbouring unit squares is at most 5. The largest and the smallest numbers on the board are 1 and 100. They can be joined by a chain of at most 19 neighbours unit squares as depicted in the figure below.

Denote by

$$a_1 = 1, a_2, a_3, \ldots, a_k = 100, \ k \leq 19$$

the numbers written in each of the neighbouring unit squares. By the triangle inequality we then have

$$
\begin{aligned}
99 \;=\; & |a_k - a_1| = |(a_k - a_{k-1}) + (a_{k-1} - a_{k-2}) + \cdots + (a_2 - a_1)| \\
\leq\; & |a_k - a_{k-1}| + |a_{k-1} - a_{k-2}| + \cdots + |a_2 - a_1| \\
\leq\; & 5(k-1) \leq 5 \cdot 18 = 90,
\end{aligned}
$$

contradiction.

Problem 9. In each unit square of a $n \times n$ array we write one of the numbers 0,1 or 2. Find all possible values of n such that computing the sum of numbers on rows and columns we obtain the numbers $1, 2, \ldots, 2n$ (not necessarily in this order).

Solution. Denote by $r_1, r_2, \ldots r_n$ and $c_1, c_2, \ldots c_n$ the sums over rows and columns respectively. Then

$$r_1 + r_2 + \cdots + r_n + c_1 + c_2 + \cdots + c_n = 1 + 2 + \cdots + 2n = n(2n+1)$$

On the other hand each number on the array is counted exactly twice in the above sum, so $n(2n+1)$ is even, tat is, n is even.

Let now $n = 2k$. We show that for each $k \geq 2$ it is possible to fulfill the above property.

We fill the first $k$unit squares on the main diagonal with 1, the last k unit squares with 2. We fill the unit squares under the main diagonal with 0 and the unit squares above the main diagonal with 2.

The sum of the elements in the first k rows is $4k-1, 4k-3, \ldots, 2k+1$ and on the last k rows is $2k, 2k-2, 2k-4, \ldots, 2$.

The sum of the elements in the first k columns is $1, 3, \ldots, 2k-1$ and on the last k rows is $2k+2, 2k+4, 2k+6, \ldots, 4k$.

Problem 10. Every cell of a 200×200 table is colored black or white. It is known that the difference between the number of black and white cells on the table is 404. Prove that some 2×2 square on the table contains an odd number of black unit squares.
(Russia Math Olympiad, 2000)

Solution. Assume by contradiction that all 2×2 squares on the table contain an even number of black (and so, white squares). Let b (resp. w) be the number of black (resp.white) squares in the first column. Note that

$$b + w = 200$$

According to our assumption the second column of the table is colored either in the same way as the first column or exactly opposite to it and this property holds for any column of the table.

Denote by x be the number of columns on the table colored exactly in the same way as the first column and let y be the total number of columns colored exactly opposite to the first column. Then

$$x + y = 200$$

The number of black squares on the table is $xb + yw$ and the number of white squares is $xw + yb$. Then

$$(xb + yw) - (xw + yb) = 404$$

which implies

(2) $$(x - y)(b - w) = 404$$

On the other hand, $x + y = b + w = 200$ implies that x, y and separately b, w have the same parity (since their sum is an even number). Therefore

$$x - y = 2m, \qquad b - w = 2n$$

for some integers m, n. Using this fact in (2) we have $4mn = 404$, that is, $mn = 101$. But this is impossible since 101 is a prime number and $|m|, |n| < 100$.